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Some remarks on 'Perturbation solutions in laminar boundary theory'

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A procedure is introduced to extend the usefulness of some perturbation solutions previously presented by Libby & Fox (1963) and Fox & Libby (1964). The perturbations are now formulated about a Blasius solution with an unknown origin. This origin, an additional degree of freedom, is selected, in the spirit of local similarity, so that it will yield a better approximation to the initial profile. With this modification the basic solution will handle a much wider class of problems successfully. Numerical examples are presented to demonstrate the improved accuracy and applicability of this new scheme.

1. Introduction

Some further considerations related to the perturbation-type solutions presented by Libby & Fox (1963) and Fox & Libby (1964) (hereafter referred to as Part 1 and Part 2) are discussed here. Although not quantitatively indicated, these previous solutions have some limitations with regard to the initial-value problems that can be treated. The initial profiles for, say, the momentum equation solutions must be close to Blasius; clearly this places a restriction on the problems which can be handled with some degree of accuracy. Indeed even for the case of small deviations from the Blasius function, solutions presented in Part 1 indicate that resort had to be made to second-order solutions to obtain reasonable results.

Techniques which can improve the accuracy of such initial-value problems are investigated here. These methods make use of the fact that the origin of the Lees co-ordinates (s, η) is unknown. In the previous papers it had been assumed that this origin and that of the initial profile were coincident. It is unnecessary, however, that this be true; indeed in some cases it appears undesirable. Further, it is recognized by the parabolic nature of the equation, that the flow downstream can be completely specified by the initial profile without reference to its upstream history. The object of the present paper is the formulation of some suitable criteria permitting the determination of this origin and a discussion of the advantages of doing so.

In addition some simple applications are presented demonstrating the usefulness of this procedure. The results are compared with those previously presented in Part 1 and with those more accurate solutions, where available.

2. Analysis

The basic problem is formulated for the momentum equation; the energy equation may be treated similarly. Consider then the momentum equation describing a laminar boundary layer with a uniform external stream in terms of the Levy–Lees variable η and s (cf. Lees 1956 and Hayes & Probstein 1959) when $\rho\mu = \text{const.}$:

$$f_{\eta\eta\eta} + ff_{\eta\eta} = 2s(f_\eta f_{\eta s} - f_{\eta\eta} f_s), \tag{2.1}$$

where

$$\eta = \rho_e u_e r^j (2s)^{-\frac{1}{2}} \int_0^y (\rho/\rho_e)\, dy, \tag{2.2}$$

$$s - s_i = \int_{x_i}^x \rho_e \mu_e u_e r^{2j} dx. \tag{2.3}$$

The transformation (2.3) is in a different form from that appearing in Parts 1 and 2; here the origin s_i, appears explicitly and is, for the moment, unknown. The associated boundary and initial conditions are:

$$f(s,0) = f_\eta(s,0) = 0, \quad f_\eta(s,\infty) = 1{\cdot}0, \quad f_\eta(s_i,\eta) = F_\eta(\eta), \tag{2.4}$$

where $F_\eta(\eta)$ is a given initial profile.

The solution to (2.1) and (2.4) may be effected by proceeding as in Part 1 and assuming

$$f_\eta(s,\eta) \approx f_0' + f_{1,1\eta}(s,\eta) + \ldots, \tag{2.5}$$

where f_0 and $f_{1,1}$ satisfy the following equations, boundary, and initial conditions:

$$\left.\begin{aligned}
& f_0''' + f_0 f_0'' = 0, \\
& f_{1,1\eta\eta\eta} + f_0 f_{1,1\eta\eta} + f_0'' f_{1,1} = 2s(f_0' f_{1,1s_\eta} - f_0'' f_{1,1_s}), \\
& f_0(0) = f_0'(0) = (0), \quad f_0'(\infty) = 1, \\
& f_{1,1}(s,0) = f_{1,1_\eta}(s,0) = f_{1,1_\eta}(s,\infty) = 0, \\
& f_{1,1_\eta}(s_i,\eta) = F_\eta(\eta) - f_0'.
\end{aligned}\right\} \tag{2.6}$$

The solution to the first of (2.6) is clearly the Blasius function, while that of the second is given in Part 1 as

$$f_{1,1}(s,\eta) = \sum_n A_{1,n}(s/s_i)^{-\frac{1}{2}\lambda_{1,n}} N_{1,n}(\eta), \tag{2.7}$$

where the $A_{1,n}$ coefficients may be found by application of the orthogonality condition for $N_{1,n}(\eta)$ so that the initial conditions are satisfied; then

$$A_{1,n} = C_{1,n}^{-1} \int_0^\infty (f_0'^4/f_0'') \left[f_0'^{-1} \int_0^\eta (F_\eta - f_0')\, d\eta \right]' (N_{1,n}/f_0')'\, d\eta. \tag{2.8}$$

Consider now the determination of s_i. It appears that when $F_\eta - f_0'$ is small in some sense, i.e. when the initial profile is close to some Blasius solution with an origin to be determined, then the $A_{i,n}$ constants are small and the series represented by (2.7) converges rapidly. Thus, as a general rule, s_i is to be calculated so that $F_\eta - f_0'$ is indeed small and consequently so that the initial profile is well fitted.

It is evident that there exist many methods to satisfy this criterion provided that a suitable definition for deciding the size of $(F_\eta - f_0')$ is available. Some suggestions for the determination of s_i follow. Noting that the integral in (2.8) contains s_i implicitly, it can be required that

$$\sum_n A_{1,n}^2 = \text{minimum}.$$

This would provide a minimum norm for the series (2.7). To insure a rapid decay for large s (or x), s_i can be determined so that $A_{1,1} = 0$. For an approximate measure of s_i the boundary layer, displacement, or momentum thickness for F_η and f_0' can be equated. It is this last, engineering approach, that will be presented here, although studies are being undertaken to determine a systematic procedure for the calculation of s_i in these and similar problems.

For clarity in the following discussion consider a two-dimensional incompressible flow and determine s_i so that the displacement thicknesses of the Blasius solution and of the initial profile are equal. The initial profile can be assumed to be given as a function of y at a station $x = x_i$. A transformation $(x, y) \to (s_0, \eta_0)$ can be applied to the profile where (s_0, η_0) are known and where

$$s_0 = \int_0^x \rho_e \mu_e u_e r^{2j} dx, \tag{2.9}$$

η_0 is given by (2.2) with s replaced by s_0, and where the external properties are arbitrarily taken to be identical to those of the Blasius solution. It is pointed out that this is a convenience and a consequence of the fact that it is unnecessary to specify more than the initial profile and its location for a parabolic problem.

If the initial profile in this new co-ordinate system is denoted by $F_{\eta_0}(\eta_0)$, then for equal displacement thicknesses

$$s_i^{\frac{1}{2}} \int_0^\infty (1 - f_0') d\eta = s_{i,0}^{\frac{1}{2}} \int_0^\infty [1 - F_{\eta_0}(\eta_0)] d\eta_0, \tag{2.10}$$

where for clarity it is repeated here that s_i is the unknown origin and where $s_{i,0} = \rho_e \mu_e u_e x_i$ and is assumed known. The values of these integrals may be presumed known and thus

$$(s_i / s_{i,0}) = a^{-1}, \tag{2.11}$$

where a is a given constant. Note that if $a \equiv 1$ arbitrarily, i.e. if the origin of the perturbation solutions is chosen to be that of the initial profile, the results of Part 1 are immediately recovered. To compute the new initial profiles the stretching from η_0 to η is required; clearly it is given by $\eta = a^{\frac{1}{2}} \eta_0$. The problem is then completely defined, i.e. $F_\eta(\eta)$ and $A_{1,n}$ can be computed and the resultant decay back to Blasius determined. Note that as a final test of the usefulness of the scale factor a, inspection of the fit of the initial profile is required.†

With this specific example in mind the extension to a more general procedure is evident. The constant a is determined first by application of any appropriate

† It should be recognized that this general procedure is an application, in some sense, of the concept of local similarity (cf. Hayes & Probstein 1959), the perturbation term acting as the correction. It is suggested that this is the first attempt to treat initial-value problems by use of this technique.

scaling condition and the $A_{1,n}$ coefficients computed.† The physically interesting parameter, the skin friction, may be computed by

$$(c_f/c_{f_0}) = (s_B/s)^{\frac{1}{2}}\{1+[f_0''(0)]^{-1}\sum_n A_{1,n}(s/s_i)^{-\frac{1}{2}\lambda_{1,n}}\}. \qquad (2.12)$$

Here c_{f_0} is interpreted as the skin friction that a Blasius solution would have if initiated from $x = 0$ to the x of interest, with s_B the corresponding transformed streamwise co-ordinate. In terms of physical variables, for any a,

$$c_f/c_{f_0} = \{a\tilde{x}/[1+a(\tilde{x}-1)]\}^{\frac{1}{2}}\{1+[f_0''(0)]^{-1}\sum A_{1,n}[1+a(\tilde{x}-1)]^{-\frac{1}{2}\lambda_{1,n}}\}, \quad (2.13)$$

where $\tilde{x} \equiv (x/x_i)$. The first bracket represents the locally similar approximation while the second is the result of a perturbation correction. It is pointed out that no mention has been made here of the second-order correction, i.e. $f_{1,2}(s,\eta)$. The approximation indicated by (2.13) should be sufficient for most purposes since the deviations from the shifted Blasius solution are now presumed small and the initial profile is well matched.

3. Applications

In Part 1 the problem was considered of a two-dimensional permeable wall, followed at $x = x_i$ by an impermeable surface. Of interest is the decay of the skin friction for $x > x_i$ as indicated in (2.13). The initial profile, at the end of the porous surface, may be obtained from Low (1955) for a variety of injection rates.

Consider first $f_w = -0{\cdot}5(2)^{-\frac{1}{2}}$, the problem presented in Part 1.‡ The initial profile is indicated in figure 1 (*a*) as $F_{\eta_0}(\eta_0)$ and after application of the scaling suggested by (2.10)§ as $F_\eta(\eta)$. It will be noted that the deviation of $F_\eta(\eta)$ from f_0' is now extremely small and the fit excellent. The resultant skin-friction distribution is shown in figure 1 (*b*), where several other results of interest are also displayed. All these may be compared with the more accurate solution presented by Pallone (1961). Clearly the new solution provides results, in this case, at least as accurate as the second-order solution of Part 1. The results corresponding to only the locally similar approximation point out that this approximation alone is indeed reasonable.

The corresponding results for a larger injection rate, $-f_w = 2^{-\frac{1}{2}}$ are shown in figures 2 (*a*) and (*b*). The deviation of $F_{\eta_0}(\eta_0)$ is now very significant and, in fact, this problem cannot be treated by strict application of the techniques of Part 1; it will be discovered that the series (2.7) cannot represent the initial profile satisfactorily. However, performing the same scaling as before leads to a new $F_\eta(\eta)$ close to f_0' and to a good fit of the profile. The skin-friction results are again in reasonable agreement with the more accurate solution of Pallone.

† The appropriateness of the particular matching used will be indicated by the subsequent fit of the initial profile.

‡ It is pointed out here that the results obtained in Part 1 are for $-f_w = 0{\cdot}5\,(2)^{-\frac{1}{2}}$ and not for $-f_w = 0{\cdot}5$, as indicated therein.

§ This scaling guarantees equal displacement thickness for incompressible flows, or for the compressible case, it may be interpreted as determining s_i so that the velocity defects in the transformed plane are equal.

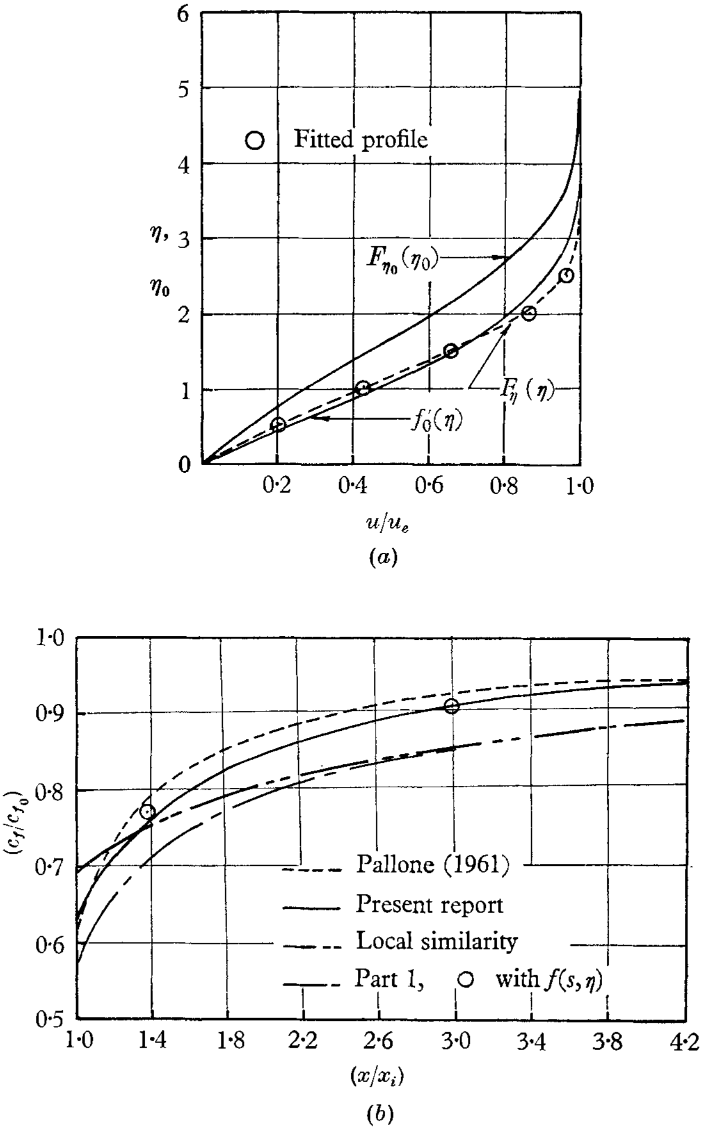

FIGURE 1. (a) Comparison of initial profiles for $-f_w = 0{\cdot}5(2)^{-\frac{1}{2}}$.
(b) Distribution of skin friction for $-f_w = 0{\cdot}5(2)^{-\frac{1}{2}}$.

To acquire some feeling for the effects of different scaling procedures consider figure 3. Here the additional skin-friction results for the preceding problems were obtained by matching momentum thicknesses and initial shear stress. These are compared with the more accurate results of Pallone and with those obtained by application of (2.10); it is seen that, for this problem, the scheme as suggested in (2.10) is the most successful. Note that these results also manifest themselves in the fit of the initial profile. That this be true is crucial; it permits a decision *a priori* as to the appropriateness of the scaling procedure.

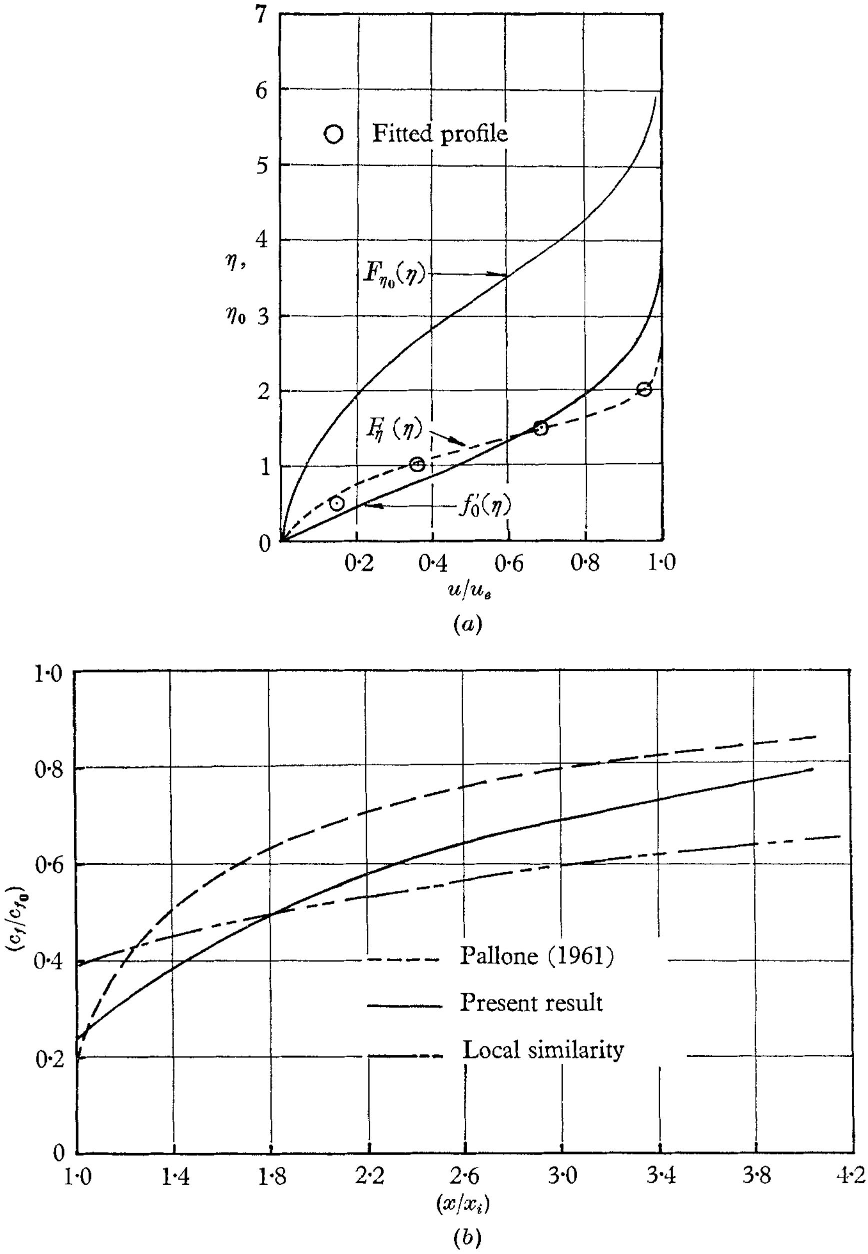

FIGURE 2. (*a*) Comparison of initial profiles for $-f_w = (2)^{-\frac{1}{2}}$. (*b*) Distribution of skin friction for $-f_w = (2)^{-\frac{1}{2}}$.

4. Concluding remarks

Based on the concept of local similarity, an extension of a perturbation technique for laminar boundary layers has been presented. This extension offers new utility to the original scheme and permits consideration of flows otherwise excluded.

It is recognized that the results presented arise from application of engineering techniques. Further studies in this general area are being undertaken to treat more rigorously the concepts presented here.

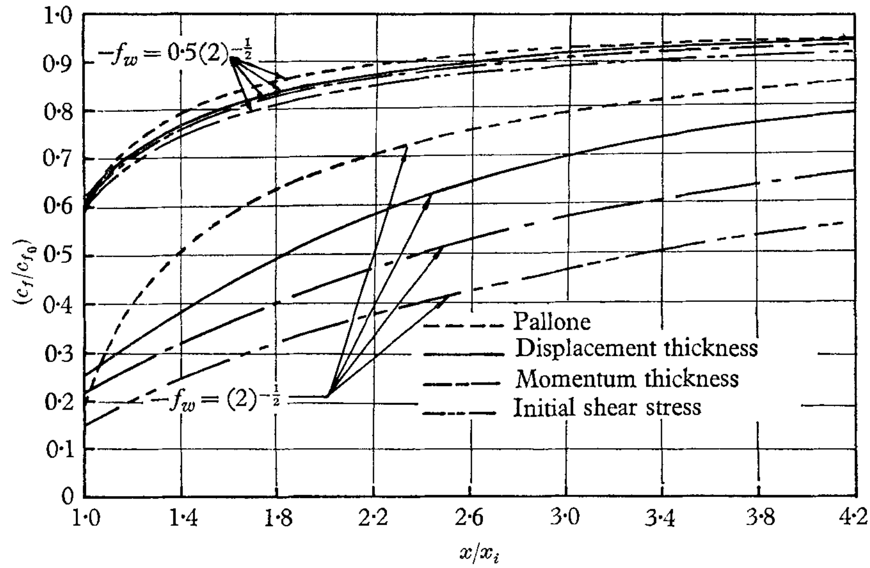

FIGURE 3. Comparison of matching techniques.

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